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Introduction to Mesoscale Modelling

Kazuo Saito
ksaito@mri-jma.go.jp

Meteorological Research Institute

1. Fundamental equations for atmosphere
2. Waves in mesoscale atmosphere
3. Classification of nonhydrostatic models
4. Numerics of JMA-NHM

1. Fundamental equations for atmosphere

Six variables which describe the state of dry atmosphere:
three velocity components, pressure, temperature and density

Prognostic equations

- Momentum equation（three wind components: u, v and w ）
- Continuity equation（pressure: p）
- Thermodynamic equation (temperature: T)

Diagnostic equation

- State equation (density: ρ)

In the case of moist atmosphere, preservation of water substances and the phase change must be considered (cloud micro-physics).

Momentum equation

- Momentum equation（three components）

$$\frac{du}{dt}+\frac{1}{\rho}\frac{\partial p}{\partial x}=dif.u$$

$$\frac{dv}{dt}+\frac{1}{\rho}\frac{\partial p}{\partial y}=dif.v$$

$$\frac{dw}{dt}+\frac{1}{\rho}\frac{\partial p}{\partial z}+g=dif.w$$

∂: partial derivative symbol

・Newton's law of motion: （Force）＝（mass × acceleration）

→

Navier-Sokes' equation for fluid：

（acceleration）＝（pressure gradient force per unit mass）

（+diffusion+gravity force for vertical direction ）

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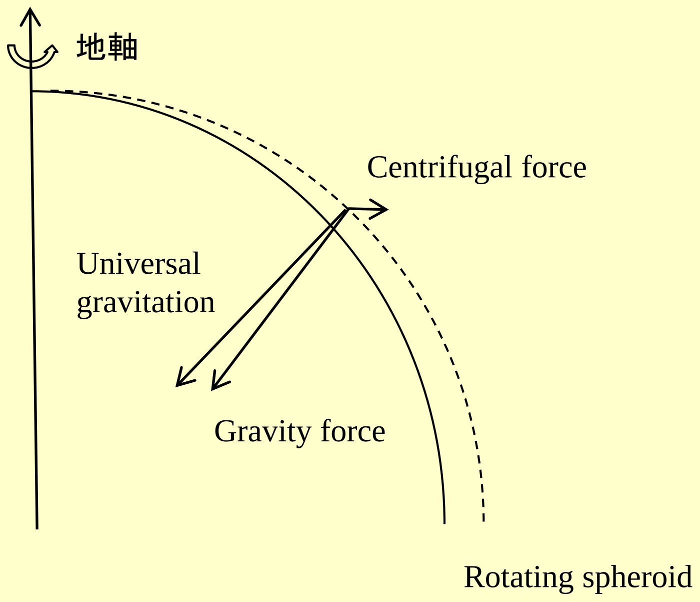
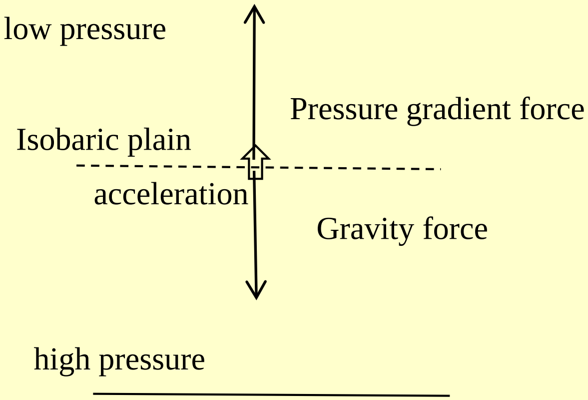

地軸
Centrifugal force
Universal
gravitation
Gravity force
Rotating spheroid

Hydrostatic equilibrium

In case the aspect ratio of the atmospheric motion is much smaller than unity, the equation for vertical motion

$$\frac{dw}{dt} + \frac{1}{\rho}\frac{\partial p}{\partial z} + g = dif.w$$



can be replaced by hydrostatic equilibrium

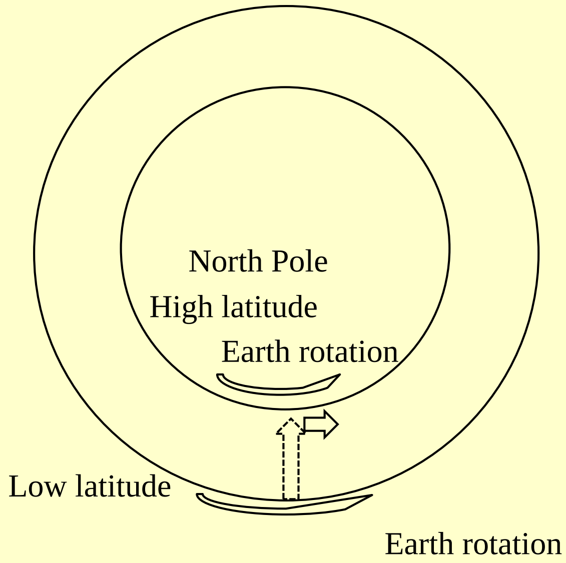
$$\frac{1}{\rho}\frac{\partial p}{\partial z} + g = 0$$

This relation equilibrium (balance) of forces between vertical pressure gradient force and gravity force.

Vertical pressure gradient force usually balances with the gravity force (hydrostatic equilibrium)

Coriolis' force

Effect of Coriolis' force is added on the earth



Northward motion shifts eastward in Northern hemisphere due to the difference of speeds of earth rotation.

In vector formulation, Coriolis' force is given by vector product of angular velocity of the earth rotation vector Ω and wind vector V:

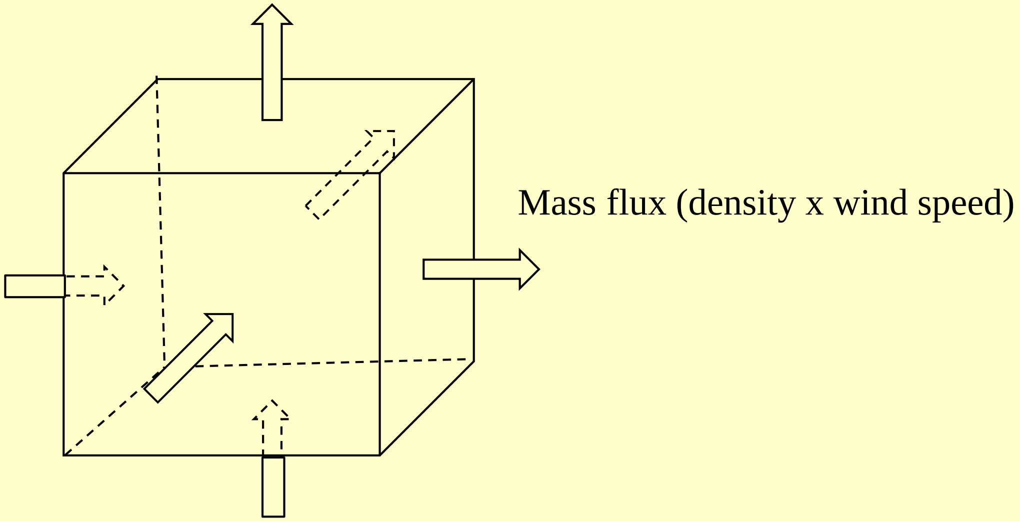
$$\frac{d\vec{V}}{dt} = (-2\vec{\Omega} \times \vec{V}) - \frac{1}{\rho}\nabla p + \vec{g} + \vec{F},$$

Continuity equation

Continuity equation（law of mass preservation）

$$\frac{\partial \rho}{\partial t}+\frac{\partial \rho u}{\partial x}+\frac{\partial \rho v}{\partial y}+\frac{\partial \rho w}{\partial z}=0$$

・・local time tendency of density=differences of mass flux through surrounding boundaries



From the following relationship,

$$\frac{\partial\rho}{\partial t}+u\frac{\partial\rho}{\partial x}+v\frac{\partial\rho}{\partial y}+w\frac{\partial\rho}{\partial z}+\rho(\frac{\rho u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z})=0$$

we obtain the continuity equation in advective form:

$$\frac{1}{\rho}\frac{d\rho}{dt}+\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

.. flow dependent time tendency of density is given by local convergence

State equation for ideal gasses

Boyle-Charles's combined law for ideal gas with a molecular weight of *m*

$$p = \rho \frac{R^*}{m} T = \frac{M}{m} \frac{R^*}{V} T$$

R^* : universal gas constant (=8.314J/mol/K)

In case of dry air (represented by subscript *d*),

by Dalton's law for partial pressure,

$$p_d = \sum_i p_i = \frac{R^*}{V} T \sum_i \frac{M_i}{m_i} = \frac{R^*}{V} T \frac{\sum_i M_i}{m_d} = \rho_d RT$$

i is index to represent gas component such as nitrogen, oxygen, and argon, and m_d the weigh-average molecular of dry air (28.966 g/mol)

$$m_d = \frac{\sum_i M_i}{\sum_i \frac{M_i}{m_i}}$$

R (=R^*/m_d) : gas constant for dry air (=287.05 J/Kg/K)

Diagnostic equation for density

State equation for dry air

$$p = \rho RT$$

Using the specific volume α *(inverse of density* ρ*)*

$$p\alpha = RT$$

If we define the non-dimensional pressure (Exner function) π
and the potential temperature θ

$$\pi = \left(\frac{p}{p_0}\right)^{R/C_p}, \quad \theta = \frac{T}{\pi}$$

（p_0=1000hPa, C_p is the specific heat of dry air in constant pressure;
7R/2=1004.7J/Kg/K）

$$\rho = \frac{p_0}{R\theta}\left(\frac{p}{p_0}\right)^{C_v/C_p}$$

where C_v is the specific heat of dry air in constant volume;

$C_v = C_p - R$=5R/2=717.6J/Kg/K

Pressure-height equation for hydrostatic atmosphere

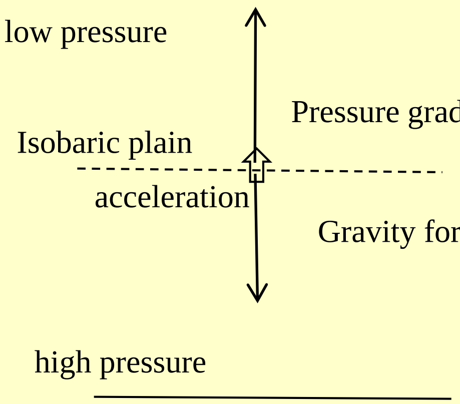
Applying the state equation to hydrostatic equilibrium

$$\frac{1}{\rho}\frac{dp}{dz}+g=0$$

$$\rightarrow \frac{1}{p}\frac{dp}{dz}=-\frac{g}{RT}$$

$$\rightarrow \frac{d}{dz}(\log p)=-\frac{g}{RT}$$

$$\rightarrow \quad p=p_0 e^{-\frac{gz}{RT_m}}$$



Veridical pressure gradient force usually balances with the gravity force (hydrostatic equilibrium)

We obtain well-known barometric height formula (pressure-height equation)

Thermodynamic equation
（Conservation of potential temperature）

First law of thermodynamics

$$dQ = dI + pd\alpha$$

$\because$ change in the internal energy of a closed system is equal to the amount of heat supplied to the system minus the amount of work done by the system on its surroundings　Let non-adiabatic heating rate Q,

$$Qdt = C_v dT + pd\alpha = (C_v + R)dT - \alpha dp$$
$$= C_P \pi d\theta$$

$$\because \ p\alpha = RT, \alpha dp = \frac{R\theta\pi}{p} d(p_0 \pi^{\frac{C_p}{R}}) = C_p \theta d\pi$$

Thus,

$$\frac{d\theta}{dt} = \frac{Q}{C_P \pi}$$

$\because$ Conservation law (prognostic equation) of potential temperature

State equation for moist air

Partial pressure of moist air

$$p = p_d + p_v = \left(\frac{M_d}{m_d} + \frac{M_v}{m_v}\right)\frac{R^*}{V}T = \left(M_d + \frac{M_v m_d}{m_v}\right)\frac{R}{V}T$$

$$= (M_d + 1.61 M_v)\frac{R}{V}T = (\rho_d + 1.61\rho_v)RT$$

$$= (\rho_a + 0.61\rho_v)RT = \rho_a(1 + 0.61 q_v)RT = \rho_a R T_v$$

ρ_a : density of moist air, T_v : virtual temperature, q_v: Specific humidity
Virtual potential temperature is defined by replacing temperature by virtual temperature in definition of potential temperature

$$\theta_v \equiv \frac{T_v}{\pi} = \frac{(1 + 0.61 q_v)T}{\pi} = (1 + 0.61 q_v)\theta$$

Thermodynamic equation for moist air

Specific heat of water vapor in constant pressure C_{pv} =1854J/Kg/K
Specific heat of water vapor in constant volume $C_{vv} = C_{pv} - R^*/m_v$=1390J/Kg/K

First law of thermodynamics for moist air is given by

$$dQ = \frac{(C_p + rC_{pv})}{1+r} dT \cong C_p(1+0.85q_v)dT$$

where r is the mixing ratio.
Likewise, specific heat of moist air in constant volume is

$$C_{vv} = \frac{(C_v + rC_{vv})}{1+r} \cong C_v(1+0.94q_v)$$

Potential temperature of moist air may be modified as

$$\theta_{moist} = T(\frac{p_0}{p})^{R_m/C_{pm}} = T(\frac{p_0}{p})^{\frac{R(1+0.61q_v)}{C_p(1+0.85q_v)}} \cong T(\frac{p_0}{p})^{\frac{R}{C_p}(1-0.24q_v)}$$

The difference between θ and θ_{moist} is less than 0.1K, and can be ignored as

$$\rho_a = \frac{p_0}{R\theta_v}(\frac{p}{p_0})^{C_v/C_p}$$

Flux form equation

Transport of water vapor

$$\frac{\partial q}{\partial t}+u\frac{\partial q}{\partial x}+v\frac{\partial q}{\partial y}+w\frac{\partial q}{\partial z}=M$$

· ·local tie tendency of *q* =advection + moisture source

Combining with the continuity equation,

$$\frac{\partial \rho}{\partial t}+\frac{\partial \rho u}{\partial x}+\frac{\partial \rho v}{\partial y}+\frac{\partial \rho w}{\partial z}=0$$

We obtain the following flux form equation,

$$\frac{\partial \rho q}{\partial t}+\frac{\partial \rho uq}{\partial x}+\frac{\partial \rho vq}{\partial y}+\frac{\partial \rho wq}{\partial z}=\rho M$$

Perturbation of the density

From state equation, perturbation of density can be divided into the following two terms:

$$\rho' = \left(\frac{p}{R\theta\pi}\right)' = \left\{\frac{p_0}{R\theta}\left(\frac{p}{p_0}\right)^{\frac{C_v}{C_p}}\right\}'$$

$$= \frac{p_0}{R}\left(\frac{p}{p_0}\right)^{\frac{C_v}{C_p}}\left(-\frac{\theta'}{\theta^2}\right) + \frac{p_0}{R\theta}\frac{C_v}{C_p}\left(\frac{p}{p_0}\right)^{\frac{C_v}{C_p}-1}\frac{p'}{p_0}$$

$$= -\rho\frac{\theta'}{\theta} + \rho\frac{C_v}{C_p}\frac{p'}{p} = -\rho\frac{\theta'}{\theta} + \frac{p'}{C_S{}^2} \qquad (A.11)$$

where

$$C_S = \sqrt{\frac{C_p}{C_v}RT} \qquad (A.12)$$

Since potential temperature is invariant for total derivation,

$$\frac{dp'}{dt} = C_S{}^2\frac{d\rho}{dt}$$

2. Waves in mesoscale atmosphere

Starting from the following 2-dimensional basic equations, we derivate three wave solutions in mesoscale atmosphere.

$$\frac{du}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \qquad (1)$$

$$\frac{dw}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g \qquad (2)$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho w}{\partial z} = 0 \qquad (3)$$

$$\frac{d\theta}{dt} = 0 \qquad (4)$$

1) Sound waves

If we linearize the equation (1)-(3) by

$u = u'$, $w = w'$, $\theta = \theta + \theta'$, $\rho = \rho_0 + \rho'$

The system becomes

$$\frac{\partial u'}{\partial t} + \frac{1}{\rho_0}\frac{\partial p'}{\partial x} = 0 \qquad (A.1'')$$

$$\frac{\partial w'}{\partial t} + \frac{1}{\rho_0}\frac{\partial p'}{\partial z} = 0 \qquad (A.2'')$$

$$\frac{\partial \rho'}{\partial t} + \rho_0\left(\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z}\right) = 0 \, (A.3'')$$

Time tendency of the density is replaced by the pressure tendency as

$$\frac{\partial p'}{\partial t} + C_S{}^2 \rho_0\left(\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z}\right) = 0 \, (A.13)$$

Eliminating *u'* and *w'* from (A.1''),(A.2''),(A.13), we obtain

$$\frac{\partial^2 p'}{\partial t^2} - C_S{}^2\left(\frac{\partial^2 p'}{\partial x^2} + \frac{\partial^2 p'}{\partial z^2}\right) = 0 \qquad (A.14)$$

Assuming the solution of *p'* as

$$p' = A\exp\{i(kx + mz - \omega t)\} \qquad (A.8)$$

We obtain the following dispersion relation:

$$\omega^2 - C_S{}^2(k^2 + m^2) = 0 \qquad (A.15)$$

Relationships between the wave number and wave length and phase speed:

$$\lambda_x = \frac{2\pi}{k}, c_x = \frac{\omega}{k}, \lambda_z = \frac{2\pi}{m}, c_z = \frac{\omega}{m}, \lambda = \frac{2\pi}{\sqrt{k^2 + m^2}}$$

Since phase speed normal to wave plain is

$$c = \frac{\omega}{\sqrt{k^2 + m^2}} = C_S$$

Cs means the sound wave speed

2) Gravity waves

If we linearize the equation (1)-(3) by

$u = u', w = w', \theta = \theta + \theta', \rho = \rho_0 + \rho'$

The system becomes

$$\frac{\partial u'}{\partial t} + \frac{1}{\rho_0}\frac{\partial p'}{\partial x} = 0 \quad (A.1')$$

$$\sigma\frac{\partial w'}{\partial t} + \frac{1}{\rho_0}\frac{\partial p'}{\partial z} = b' \quad (A.2')$$

$$\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} = 0 \quad (A.3')$$

$$\frac{\partial b'}{\partial t} + w' N^2 = 0 \quad (A.4')$$

where

$b' = g\theta'/\theta$: buoyancy

$N^2 = g/\theta \times d\theta/dz$: Brunt-Visala's frequency

$\sigma = 0$ for hydrostatic, $\sigma = 1$ for a nonhydrostatic system

Eliminating *u'* from (A.1') and (A.3'), and eliminating *b'* from (A.2') and (A.4'), we obtain the following equations for *p'* and *w'*

$$\frac{\partial^2 w'}{\partial t \partial z} - \frac{1}{\rho_0} \frac{\partial^2 p'}{\partial x^2} = 0 \qquad (A.5)$$

$$\sigma \frac{\partial^2 w'}{\partial t^2} + \frac{1}{\rho_0} \frac{\partial^2 p'}{\partial t \partial z} + w' N^2 = 0 \quad (A.6)$$

Further eliminating *p'* we obtain the following elliptic equation for *w'*

$$\frac{\partial^2}{\partial t^2}\left(\sigma \frac{\partial^2 w'}{\partial x^2} + \frac{\partial^2 w'}{\partial z^2}\right) + N^2 \frac{\partial^2 w'}{\partial x^2} = 0 \, (A.7)$$

If we assume solution of *w'* as

$$w' = A\exp\{i(kx + mz - \omega t)\} \qquad (A.8)$$

Relationships between the wave number and wave length and phase speed are:

$$\lambda_x = \frac{2\pi}{k}, c_x = \frac{\omega}{k}, \lambda_z = \frac{2\pi}{m}, c_z = \frac{\omega}{m}$$

$$\lambda = \frac{2\pi}{\sqrt{k^2 + m^2}}, c = \frac{\omega}{\sqrt{k^2 + m^2}} \qquad (A.9)$$

Dispersion relation for gravitiy waves is

$$\omega^2 = \frac{k^2 N^2}{\sigma k^2 + m^2} \qquad (A.10)$$

In hydrostatic system (σ=0 or $m>>k$),

$$\omega = kN/m, \; C_x = N/m$$

Phase speed for horizontal direction is independent from the wave number, thus hydrostatic gravity waves do not have dispersibility horizontally.

In nonhydrostatic case (σ=1 and $m \sim k$),

ω becomes smaller than N.

In the limit of $k=\infty$, (in case of vertical motion only)
magnitude of ω becomes N

3) Mountain waves

Steady state linear mountain wave is a special case where upward phase velocity of internal gravity wave is in equilibrium with the ambient wind *U*.

If we linearize the equation (1)-(3) by

$$u=U+u', \; w=w', \; \theta=\theta+\theta', \; \rho=\rho_0$$

and assuming the steady state (time tendency is zero), the system becomes

$$U\frac{\partial u'}{\partial x}+\frac{1}{\rho_0}\frac{\partial p'}{\partial x}=0 \quad (B1)$$

$$\sigma U\frac{\partial w'}{\partial t}+\frac{1}{\rho_0}\frac{\partial p'}{\partial z}=b' \quad (B2)$$

$$\frac{\partial u'}{\partial x}+\frac{\partial w'}{\partial z}=0 \quad (B3)$$

$$U\frac{\partial b'}{\partial x}+w'N^2=0 \quad (B4)$$

Eliminating *u'* from (B1) and (B3), and *b'* from (B2) and (B4), we obtain the following equations for *p'* and *w'*

$$U\frac{\partial w'}{\partial z}-\frac{1}{\rho_0}\frac{\partial p'}{\partial x}=0 \quad (B.5)$$

$$\sigma\frac{\partial^2 w'}{\partial x^2}+\frac{U}{\rho_0}\frac{\partial^2 p'}{\partial x\partial z}+w'N^2=0 \ (B.6)$$

Further eliminating *p'* we obtain the following elliptic equation for *w'*

$$U^2\left(\sigma\frac{\partial^2 w'}{\partial x^2}+\frac{\partial^2 w'}{\partial z^2}\right)+N^2 w'=0 \quad (B.7)$$

If we assume solution of *w'* as in (A.8), we obtain the following (famous) Long's (1952) equation:

$$\frac{\partial^2 w'}{\partial z^2}+(l^2-\sigma k^2)w'=0 \quad (B.8)$$

where $l=N/U$ is the scorer parameter.

In hydrostatic system (σ=0 or $m>>k$), the solution is periodic waves with the vertical wave length $2\pi/l$ without the dispersibility horizontally.

In nonhydrostatic case (σ=1 and $m \sim k$), characteristic of the solution depends on the magnitude relation of l and k.

In case $k^2 > l^2$,

the wave amplitude decreases exponentially in vertical (external wave).

In case $k^2 < l^2$,

the waves become periodic ones with the following vertical wave length (internal wave).

$$\lambda_Z = \frac{2\pi}{\sqrt{l^2 - k^2}} = 0 \quad (B.8)$$

The real solution of mountain waves can be obtained by superposition of several waves with the corresponding wave numbers.

3’) Three dimensional mountain waves

In case of 3-dimensional mountain waves, the set of equation becomes

$$U\frac{\partial u'}{\partial x}+\frac{1}{\rho_0}\frac{\partial p'}{\partial x}=0 \quad (B1)$$

$$U\frac{\partial v'}{\partial x}+\frac{1}{\rho_0}\frac{\partial p'}{\partial y}=0 \quad (B1')$$

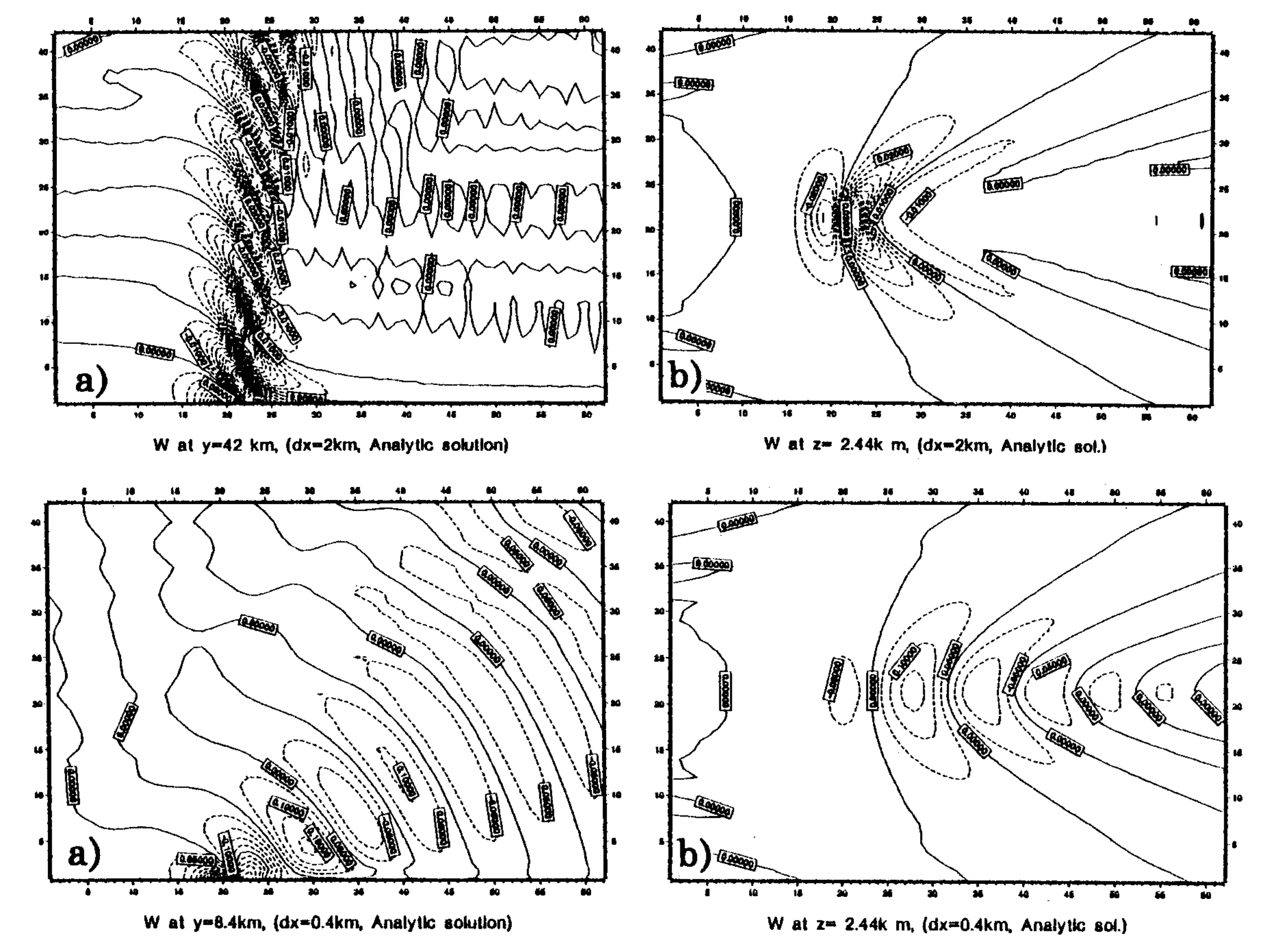
$$\sigma U\frac{\partial w'}{\partial t}+\frac{1}{\rho_0}\frac{\partial p'}{\partial z}=b' \quad (B2)$$

$$\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}+\frac{\partial w'}{\partial z}=0 \quad (B3')$$

$$U\frac{\partial b'}{\partial x}+w'N^2=0 \quad (B4)$$

and the resultant equation for w’ becomes

$$U^2[\frac{\partial^2}{\partial x^2}\{\sigma(\frac{\partial^2 w'}{\partial x^2}+\frac{\partial^2 w'}{\partial y^2})+\frac{\partial^2 w'}{\partial z^2}\}]+N^2(\frac{\partial^2 w'}{\partial x^2}+\frac{\partial^2 w'}{\partial y^2})=0 \quad (B.7')$$



Example of linear mountain waves over a 3-dimensional ax-symmetric bell-shaped mountain (*h*=100 m) in the stable atmosphere (dθ/dz=3 K/km) of *U*=8m/s.

Upper) hydrostatic case (half width *a* = 6 km), Lower) nonhydrostatic case *a* = 1.2 km

Left) Vertical cross-section through mountain top, Right) Horizontal plain at 2.44 km AGL.

3. Classification of nonhydrostatic models
1) classification by the continuity equation

Basic equations

$$\frac{d\mathbf{v}}{dt} = -\frac{1}{\rho}\nabla p - g\mathbf{k}, \qquad (1.1)$$

$$\frac{d\rho}{dt} = -\rho\nabla\cdot\mathbf{v}, \qquad (1.2)$$

$$\frac{d\theta}{dt} = \frac{Q}{C_P\pi}, \qquad (1.3)$$

$$p = \rho RT, \qquad (1.4)$$

a. Anelastic model

The anelastc (AE) model removes sound waves from solutions by a scale analysis. Field variables are divided into the time independent horizontal uniform reference state *f (z)* and its perturbation *f' (x, y, z, t)* as

$$p = \overline{p} + p', \rho = \overline{\rho} + \rho', \theta = \overline{\theta} + \theta' \quad (1.12)$$

Substituting the reference density in the momentum and continuity equations, we obtain governing equations of the anelastic model

$$\frac{\partial \overline{\rho}\mathbf{v}}{\partial t} + \mathbf{Adv} + \nabla p' = -\rho' g\mathbf{k}, \qquad (1.13)$$

$$\nabla \cdot (\overline{\rho}\mathbf{v}) = 0, \qquad (1.14)$$

Momentum equation may be rewritten as

$$\frac{\partial \overline{\rho}\mathbf{v}}{\partial t} + \mathbf{Adv} + \nabla p' + \frac{p'}{\overline{\overline{C_s}}^2} g\mathbf{k} = \overline{\rho}\frac{\theta'}{\overline{\overline{\theta}}} g\mathbf{k}. \qquad (1.16)$$

Because the density perturbation can be divided into the perturbation of potential temperature and pressure as

$$\rho' = (\frac{p}{R\theta\pi})' = \{\frac{p_0}{R\theta}(\frac{p}{p_0})^{\frac{C_v}{C_p}}\}'$$

$$= \frac{p_0}{R}(\frac{p}{p_0})^{\frac{C_v}{C_p}}(-\frac{\theta'}{\theta^2}) + \frac{p_0}{R\theta}\frac{C_v}{C_p}(\frac{p}{p_0})^{\frac{C_v}{C_p}-1}\frac{p'}{p_0}$$

$$= -\rho\frac{\theta'}{\theta} + \rho\frac{C_v}{C_p}\frac{p'}{p} = -\rho\frac{\theta'}{\theta} + \frac{p'}{C_S{}^2} \qquad (1.15)$$

b. Quasi compressible model

The quasi-compressible model considers the compressibility of air and predicts the pressure from divergence, while the reference density is used for momentum equations.

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\bar{\rho} \mathbf{v}) = 0. \qquad (1.20)$$

Using the relation of

$$\rho' = -\rho \frac{\theta'}{\theta} + \frac{p'}{C_S{}^2},$$

following pressure equation is obtained

$$\frac{\partial p}{\partial t} = -\overline{C_S{}^2} \{\nabla \cdot (\bar{\rho} \mathbf{v}) - \frac{\bar{\rho}}{\bar{\theta}} \frac{\partial \theta}{\partial t}\}, \qquad (1.21)$$

if the Exner function is used to represent the pressure, the pressure equation is given by

$$\frac{d\pi}{dt} = -\frac{R\pi}{C_v} (\nabla \cdot \mathbf{v}) + \frac{C_S{}^2}{C_p \bar{\theta}^2} \frac{d\theta}{dt}. \qquad (1.21')$$

Momentum equation is same as in the anelastic equation; (1.16).

c. Fully compressible model

The fully compressible model uses basic equations without the linearization by the reference atmosphere

$$\frac{d\mathbf{v}}{dt} = -\frac{1}{\rho}\nabla p - g\mathbf{k}, \qquad (1.1)$$

$$\frac{d\rho}{dt} = -\rho\nabla\cdot\mathbf{v}, \qquad (1.2)$$

$$p = \rho RT, \qquad (1.4)$$

Since the fully compressible model includes sound waves in its solutions and allows the time change of the density, careful attention must be paid on computation of sound waves and computational accuracy in the finite discretization.

Table 1. Classification of nonhydrostatic models and their behaviors.

	Adiabatic expansion			Constant volume heating		Constant pressure heating	
Classification	Pressure	Density	Volume	Pressure	Density	Volume	Density
Anelastic (AE)	impossible			invariant	invariant	invariant	invariant
Quasi-compressible without the thermal expansion term	decrease	invariant	increase	invariant	invariant	invariant	invariant
Quasi-compressible with the thermal expansion term	decrease	invariant	increase	increase	invariant	increase	invariant
Fully-compressible without the thermal expansion term	decrease	decrease	increase	invariant	decrease	invariant	decrease
Fully-compressible with the thermal expansion term	decrease	decrease	increase	increase	invariant	increase	decrease

$$\frac{\partial p}{\partial t} = -C_S{}^2\{\nabla \cdot (\rho \mathbf{v}) \underline{- \frac{\rho}{\theta}\frac{\partial \theta}{\partial t}}\}, \qquad (1.21)$$

thermal expansion term

2) treatment of sound waves

Nonhydrostatic models can be classified further by the treatment of sound waves.

a. AE scheme

Taking total divergence of momentum equations, the following 3-dimensional Poisson-type pressure diagnostic equation is obtained.

$$\nabla^2 P + \frac{\partial}{\partial z}(hP) = FP.AE\,, \qquad (1.25)$$

where *P=p'*, and $h=g/C_s^2$, and *r.h.s.* is the forcing term by divergence of advection and buoyancy terms.

$$FP.AE = \frac{\partial}{\partial x}(Dif.\overline{\rho}u - Adv.\overline{\rho}u) + \frac{\partial}{\partial y}(Dif.\overline{\rho}v - Adv.\overline{\rho}v)$$

$$+\frac{\partial}{\partial z}(\overline{\rho}\frac{\theta'}{\overline{\theta}}g + Dif.\overline{\rho}w - Adv.\overline{\rho}w) + \frac{1}{2\Delta t}DIV^{t-\Delta t} \qquad (1.25)$$

Here, the last term of r.h.s. (*DIV)* is the divergence at the time step $t\text{-}\Delta t$, which is required for computational stability to adjust the divergence zero at next time step (Clark, 1977).

b. HE-VI scheme

The HE-VI (horizontally explicit vertically implicit) scheme treats sound waves implicitly only for the vertical direction. This scheme is often referred as the *split-explicit* method because sound waves are treated in a short time step Δτ while low frequency modes and physical processes are treated in a long time step Δt.

Following 1-dimensional Helmholtz pressure equation is obtained

$$\frac{\partial^2 \overline{P^\tau}}{\partial z^2} + \frac{\partial}{\partial z}(h\overline{P^\tau}) + e'\overline{P^\tau} = FP.HE \,, \qquad (1.44)$$

The pressure equation (1.44) is formally similar to the pressure equation of the HI-VI scheme (1.36) except the Laplacian in the pressure equation is vertically 1-dimensional.

For example, if we define time tendency term of *A* as

$$\delta_\tau A \equiv \frac{A^{\tau+\Delta\tau} - A^{\tau}}{\Delta\tau} \qquad (1.34)$$

Momentum equations and the pressure equations may be written as

$$\delta_\tau U + \frac{\partial P}{\partial x} = FU \qquad (1.26')$$

$$\delta_\tau V + \frac{\partial P}{\partial y} = FV \qquad (1.27')$$

$$\delta_\tau W + \left(\frac{\partial}{\partial z} + \frac{g}{C_S{}^2}\right)P^{\tau+\Delta\tau} = FW \qquad (1.28')$$

$$\delta_\tau P + C_S{}^2\left(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z}\right)^{\tau+\Delta\tau} = FP.E \quad (1.29')$$

$$U^{\tau+\Delta\tau} = U^{\tau} + \Delta\tau(FU - \frac{\partial P}{\partial x}) \quad (1.35)$$

$$V^{\tau+\Delta\tau} = V^{\tau} + \Delta\tau(FV - \frac{\partial P}{\partial y}) \quad (1.36)$$

$$W^{\tau+\Delta\tau} = W^{\tau} + \Delta\tau\{FW - (\frac{\partial}{\partial z} + \frac{g}{C_S{}^2})P^{\tau+\Delta\tau}\} \quad (1.37)$$

$$P^{\tau+\Delta\tau} = P^{\tau} + \Delta\tau\{FP.E - C_S{}^2(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z})^{\tau+\Delta\tau}\} \quad (1.38)$$

Solving for $P^{\tau+\Delta t}$

$$\frac{\partial^2}{\partial z^2}P^{\tau+\Delta\tau} + \frac{\partial}{\partial z}(\frac{g}{C_S{}^2}P^{\tau+\Delta\tau}) - \frac{1}{(C_S\Delta\tau)^2}P^{\tau+\Delta\tau} = FP.HE \quad (1.39)$$

$$FP.HE = -\frac{1}{\Delta\tau}\frac{FP.E}{C_S{}^2} + \frac{\partial}{\partial z}FW$$

$$+ \frac{1}{\Delta\tau}\{(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y})^{\tau+\Delta\tau} + \frac{\partial W^{\tau}}{\partial z}\} - \frac{1}{(C_S\Delta\tau)^2}P^{\tau} \quad (1.40)$$

c. HI-VI scheme

The HI-VI (horizontally implicit vertically implicit) scheme treats sound waves implicitly for both vertical and horizontal directions. This scheme, often referred as the *semi-implicit* method.

The following 3-dimensional Helmholtz-type pressure equation is used

$$\nabla^2 P' + \frac{\partial}{\partial z}(hP') + eP' = FP.HI \ , \qquad (1.36)$$

where *P'* is defined by

$$P' = \overline{P^t} - P^t . \qquad (1.31)$$

Above pressure equation (1.36) is formally similar to the anelastic pressure equation (1.25).

For example, if we define time tendency term of A as

$$\delta_t A \equiv \frac{A^{t+\Delta t} - A^{t-\Delta t}}{2\Delta t} = \frac{\Delta^2 A}{(1+\alpha)\Delta t} + \frac{(A^t - A^{t-\Delta t})}{(1+\alpha)\Delta t} \qquad (1.34')$$

where the term –t is weight average of A at $t\text{-}\Delta t$ and $t\text{+}\Delta t$

$$\overline{A}^t = \frac{1+\alpha}{2} A^{t+\Delta t} + \frac{1-\alpha}{2} A^{t-\Delta t} \qquad (1.41')$$

Momentum equations and the pressure equations may be written as

$$\delta_t U + \frac{\partial \overline{P}^t}{\partial x} = FU \qquad (1.26'')$$

$$\delta_t U + \frac{\partial \overline{P}^t}{\partial y} = FV \qquad (1.27'')$$

$$\delta_t W + \left(\frac{\partial}{\partial z} + \frac{g}{C_S{}^2}\right)\overline{P}^t = FW \qquad (1.28'')$$

$$\delta_t P + C_S{}^2\left(\frac{\partial \overline{U}^t}{\partial x} + \frac{\partial \overline{V}^t}{\partial y} + \frac{\partial \overline{W}^t}{\partial z}\right) = FP.E \; (1.29'')$$

If we define Δ^2 as an operator

$$\Delta^2 A \equiv \overline{A}^t - A^t \qquad (1.42)$$

Momentum equations and the pressure equations (1.26’’) ∼(1.29’’)may be written as

$$\frac{\Delta^2 U}{(1+\alpha)\Delta t} + \frac{\partial \Delta^2 P}{\partial x} = FU - \frac{U^t - U^{t-\Delta t}}{(1+\alpha)\Delta t} - \frac{\partial P^t}{\partial x} \quad (1.43)$$

$$\frac{\Delta^2 V}{(1+\alpha)\Delta t} + \frac{\partial \Delta^2 P}{\partial y} = FV - \frac{V^t - V^{t-\Delta t}}{(1+\alpha)\Delta t} - \frac{\partial P^t}{\partial y} \quad (1.44)$$

$$\frac{\Delta^2 W}{(1+\alpha)\Delta t} + \left(\frac{\partial}{\partial z} + \frac{g}{C_S{}^2}\right)\Delta^2 P = FW - \frac{W^t - W^{t-\Delta t}}{(1+\alpha)\Delta t} - \left(\frac{\partial}{\partial z} + \frac{g}{C_S{}^2}\right)P^t \quad (1.45)$$

$$\frac{\Delta^2 P}{(1+\alpha)\Delta t} + C_S{}^2\left(\frac{\partial \Delta^2 U}{\partial x} + \frac{\partial \Delta^2 V}{\partial y} + \frac{\partial \Delta^2 W}{\partial z}\right)$$

$$= FP.E + \frac{P^t - P^{t-\Delta t}}{(1+\alpha)\Delta t} + C_S{}^2\left(\frac{\partial U^t}{\partial x} + \frac{\partial V^t}{\partial y} + \frac{\partial W^t}{\partial z}\right) \quad (1.46)$$

We obtain the following elliptic tendency equation for $\Delta^2 P$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\Delta^2 P+\frac{\partial}{\partial z}\left(\frac{g}{C_S{}^2}\Delta^2 P\right)-\frac{1}{(C_S\Delta t)^2(1+\alpha)^2}\Delta^2 P = FP.HI \qquad (1.47)$$

where

$$FP.HI = \left(\frac{\partial FU'}{\partial x}+\frac{\partial FV'}{\partial y}+\frac{\partial FW'}{\partial z}\right)-\frac{FP'}{C_S{}^2(1+\alpha)\Delta t} \qquad (1.48)$$

and *FU’*,*FV’*,*FW’*,*FP’* are *r.h.s.* of (1.43)～(1.46), respectively.

2.4. Characteristics in three methods

Method	Accuracy	Comput ational robust- ness	Pressure equation	Scalability in parallel computati on	Efficiency in large scale computati on	Compatibi lity with semi-Lagrangia n scheme	Compatibi lity with spectral method
AE	Anelastic approxi- mation	Good	3D Poisson	Depends on elliptic solver	Depends on elliptic solver	Good	Good
HI-VI	Good	Fair	3D Helmholtz	Depends on elliptic solver	Depends on elliptic solver	Good	Good
HE-VI	Good	Fair	1D Helmholtz	Good	Good	Sound waves exist	Fair

4. Numerics of JMA-NHM

Basic Equations

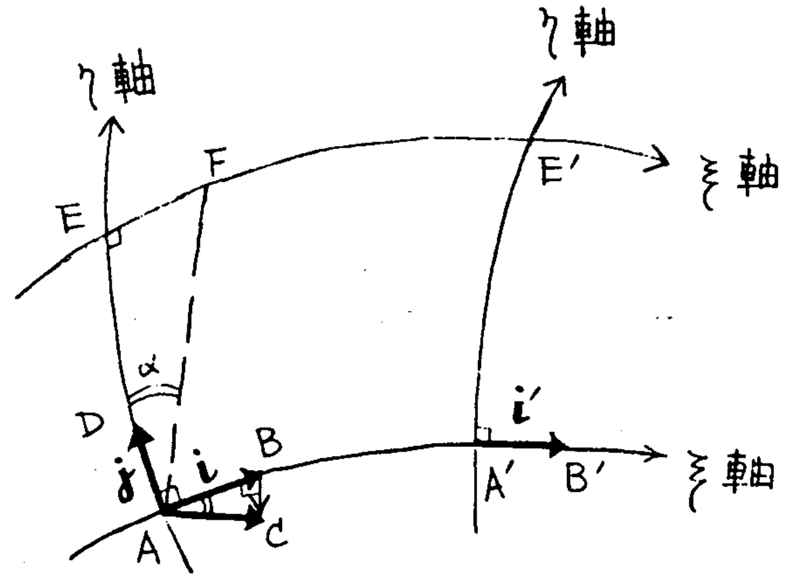
The governing basic equations of the model consist of the flux form equations on the spherical curvilinear orthogonal coordinate.
Let m_1 and m_2 be the map factors in the ξ (x) and η (y) directions,

$$ds_1 = \frac{d\xi}{m_1}, \quad ds_2 = \frac{d\eta}{m_2}, \quad ds_3 = dz,$$

$$V = \vec{i}u + \vec{j}v + \vec{k}w,$$

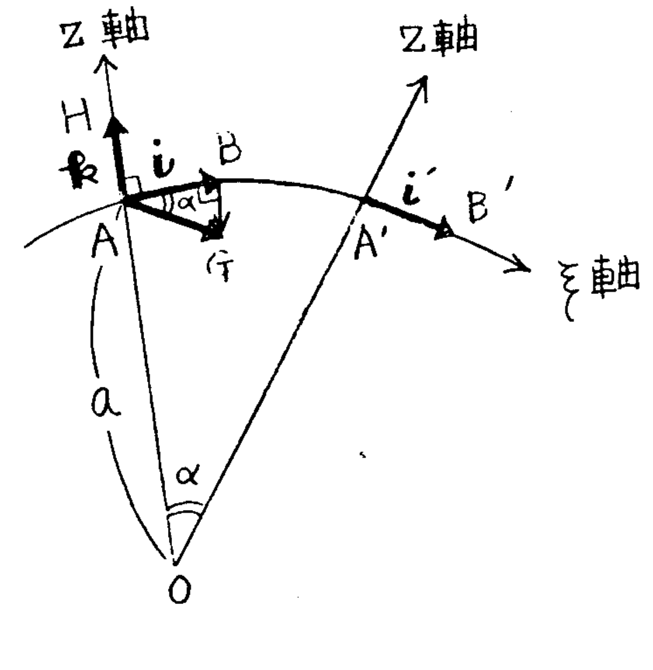
$$u = \frac{ds_1}{dt} = \frac{1}{m_1}\frac{d\xi}{dt}, \quad v = \frac{ds_2}{dt} = \frac{1}{m_2}\frac{d\eta}{dt},$$

$$w = \frac{ds_3}{dt} = \frac{dz}{dt}$$



第 A 1.1図　$z = const$ の面の曲線座標系

Momentum equation is



第 A 1.2 図 ξ − z 断面図

$$\frac{d\vec{V}}{dt} = -2\vec{\Omega}\times\vec{V} - \frac{1}{\rho}\nabla p + \vec{g} + \vec{F},$$

where

$$\nabla = \vec{i}m\frac{\partial}{\partial\xi} + \vec{j}n\frac{\partial}{\partial\eta} + \vec{k}\frac{\partial}{\partial z},$$

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{d\xi}{dt}\frac{\partial}{\partial\xi} + \frac{d\eta}{dt}\frac{\partial}{\partial\eta} + \frac{dz}{dt}\frac{\partial}{\partial z} = \frac{\partial}{\partial t} + mu\frac{\partial}{\partial\xi} + nv\frac{\partial}{\partial\eta} + w\frac{\partial}{\partial z},$$

$$\frac{d\vec{V}}{dt} = \vec{i}\frac{du}{dt} + \vec{j}\frac{dv}{dt} + \vec{k}\frac{dw}{dt} + u\frac{d\vec{i}}{dt} + v\frac{d\vec{j}}{dt} + w\frac{d\vec{k}}{dt}$$

$$= \vec{i}\frac{du}{dt} + \vec{j}\frac{dv}{dt} + \vec{k}\frac{dw}{dt} + u(mu\frac{\partial\vec{i}}{\partial\xi} + nv\frac{\partial\vec{i}}{\partial\eta}) + v(mu\frac{\partial\vec{j}}{\partial\xi} + nv\frac{\partial\vec{j}}{\partial\eta}) + w(mu\frac{\partial\vec{k}}{\partial\xi} + nv\frac{\partial\vec{k}}{\partial\eta})$$

Change of unit vectors along each coordinates are

$$\frac{\partial \vec{i}}{\partial \xi} = -\vec{j}n\frac{\partial}{\partial \eta}(\frac{1}{m}) - \vec{k}\frac{1}{am}, \qquad \frac{\partial \vec{i}}{\partial \eta} = \vec{j}m\frac{\partial}{\partial \xi}(\frac{1}{n}),$$

$$\frac{\partial \vec{j}}{\partial \xi} = \vec{i}n\frac{\partial}{\partial \eta}(\frac{1}{m}), \qquad \frac{\partial \vec{j}}{\partial \eta} = -\vec{i}m\frac{\partial}{\partial \xi}(\frac{1}{n}) - \vec{k}\frac{1}{an},$$

$$\frac{\partial \vec{k}}{\partial \xi} = -\vec{i}\frac{1}{am}, \qquad \frac{\partial \vec{k}}{\partial \eta} = -\vec{j}\frac{1}{an}$$

where a is radius of earth.

Curvature term is

$$\overrightarrow{M} = u(mu\frac{\partial \vec{i}}{\partial \xi} + nv\frac{\partial \vec{i}}{\partial \eta}) + v(mu\frac{\partial \vec{j}}{\partial \xi} + nv\frac{\partial \vec{j}}{\partial \eta}) + w(mu\frac{\partial \vec{k}}{\partial \xi} + nv\frac{\partial \vec{k}}{\partial \eta})$$

$$= \vec{i}mnv\{u\frac{\partial}{\partial \eta}(\frac{1}{m}) - v\frac{\partial}{\partial \xi}(\frac{1}{n})\} + \vec{i}\frac{wu}{a}$$

$$+ \vec{j}mnu\{-u\frac{\partial}{\partial \eta}(\frac{1}{m}) + v\frac{\partial}{\partial \xi}(\frac{1}{n})\} + \vec{j}\frac{wv}{a} - \vec{k}\frac{u^2 + v^2}{a}$$

Continuity equation is

$$\frac{d\rho}{dt} + \rho \nabla \vec{V} = 0$$

$$\nabla \vec{V} = (\vec{i} m \frac{\partial}{\partial \xi} + \vec{j} n \frac{\partial}{\partial \eta} + \vec{k} \frac{\partial}{\partial z})(\vec{i} u + \vec{j} v + \vec{k} w)$$

$$= m \frac{\partial u}{\partial \xi} + \vec{i} m (v \frac{\partial \vec{j}}{\partial \xi} + w \frac{\partial \vec{k}}{\partial \xi}) + n \frac{\partial v}{\partial \eta} + \vec{j} n (u \frac{\partial \vec{i}}{\partial \eta} + w \frac{\partial \vec{k}}{\partial \eta}) + \frac{\partial w}{\partial z}$$

$$= m \frac{\partial u}{\partial \xi} + mnv \frac{\partial}{\partial \eta} (\frac{1}{m}) + \frac{w}{a} + n \frac{\partial v}{\partial \eta} + mnu \frac{\partial}{\partial \xi} (\frac{1}{n}) + \frac{w}{a} + \frac{\partial w}{\partial z}$$

$$= mn \{ \frac{\partial}{\partial \xi} (\frac{u}{n}) + \frac{\partial}{\partial \eta} (\frac{v}{m}) \} + \frac{\partial w}{\partial z} + \frac{2w}{a}$$

Conformal projections

Set $n=m$ in the curvilinear orthogonal coordinates equations , and notate ξ, η by x, y,

$$\frac{du}{dt} = Cor_1 + Crv_1 - \frac{1}{\rho} m \frac{\partial p}{\partial x} + Dif_1,$$

$$\frac{dv}{dt} = Cor_2 + Crv_2 - \frac{1}{\rho} m \frac{\partial p}{\partial y} + Dif_2,$$

$$\frac{dw}{dt} = Cor_3 + Crv_3 - \frac{1}{\rho} \frac{\partial p}{\partial z} - g + Dif_3,$$

where

$$Cor_1 = 2\Omega \sin\varphi v - 2\Omega \cos\varphi \cos c\Delta\lambda w$$

$$Cor_2 = -2\Omega \cos\varphi \sin c\Delta\lambda w - 2\Omega \sin\varphi u$$

$$Cor_3 = 2\Omega \cos\varphi \cos c\Delta\lambda u + 2\Omega \cos\varphi \sin c\Delta\lambda v$$

$$Crv_1 = m^2 v\{v \frac{\partial}{\partial x}(\frac{1}{m}) - u \frac{\partial}{\partial y}(\frac{1}{m})\} - \frac{uw}{a}$$

$$Crv_2 = m^2 u\{u \frac{\partial}{\partial y}(\frac{1}{m}) - v \frac{\partial}{\partial x}(\frac{1}{m})\} - \frac{vw}{a}$$

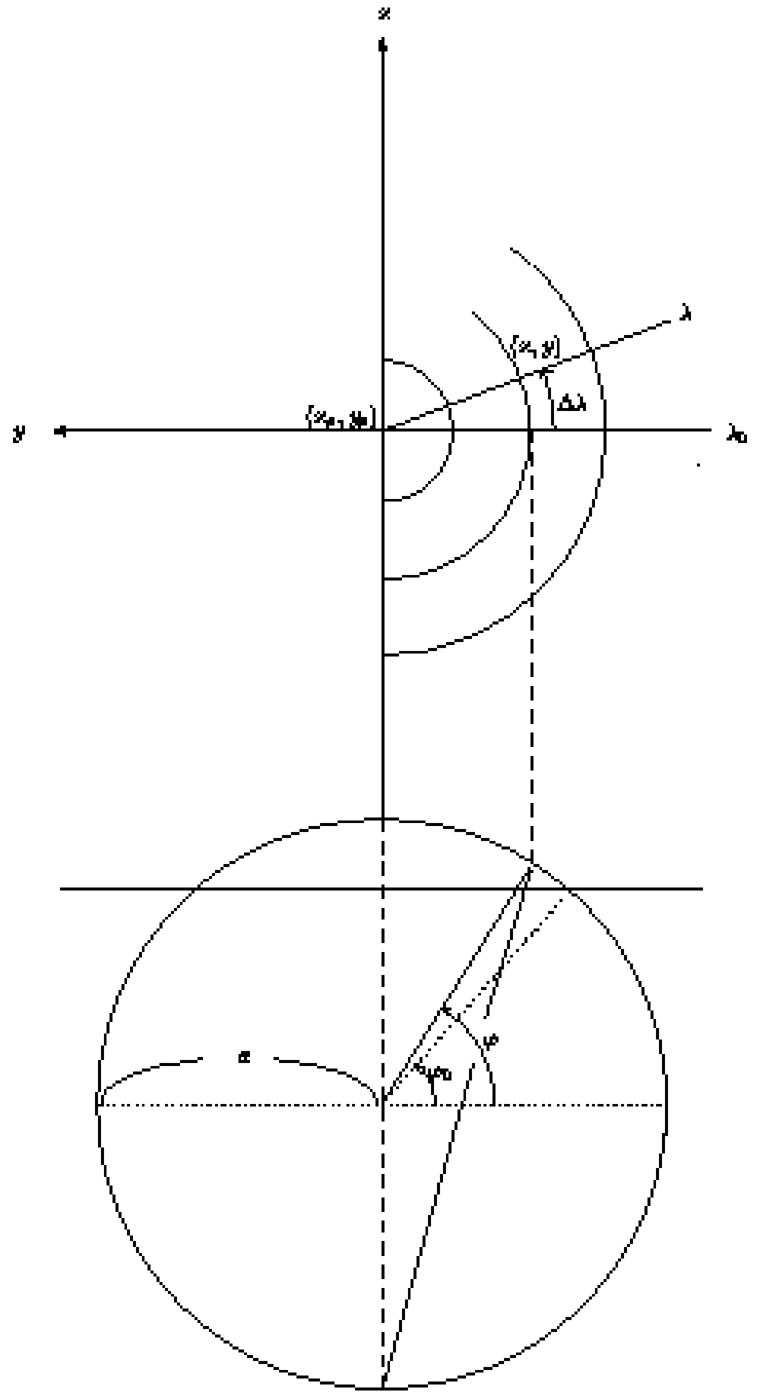
$$Crv_3 = \frac{u^2 + w^2}{a}$$

1) Polar stereo projection

$$c = 1,$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_p + ma\cos\varphi\sin\Delta\lambda \\ y_p - ma\cos\varphi\cos\Delta\lambda \end{pmatrix},$$

$$m = \frac{1+\sin\varphi_0}{1+\sin\varphi},$$



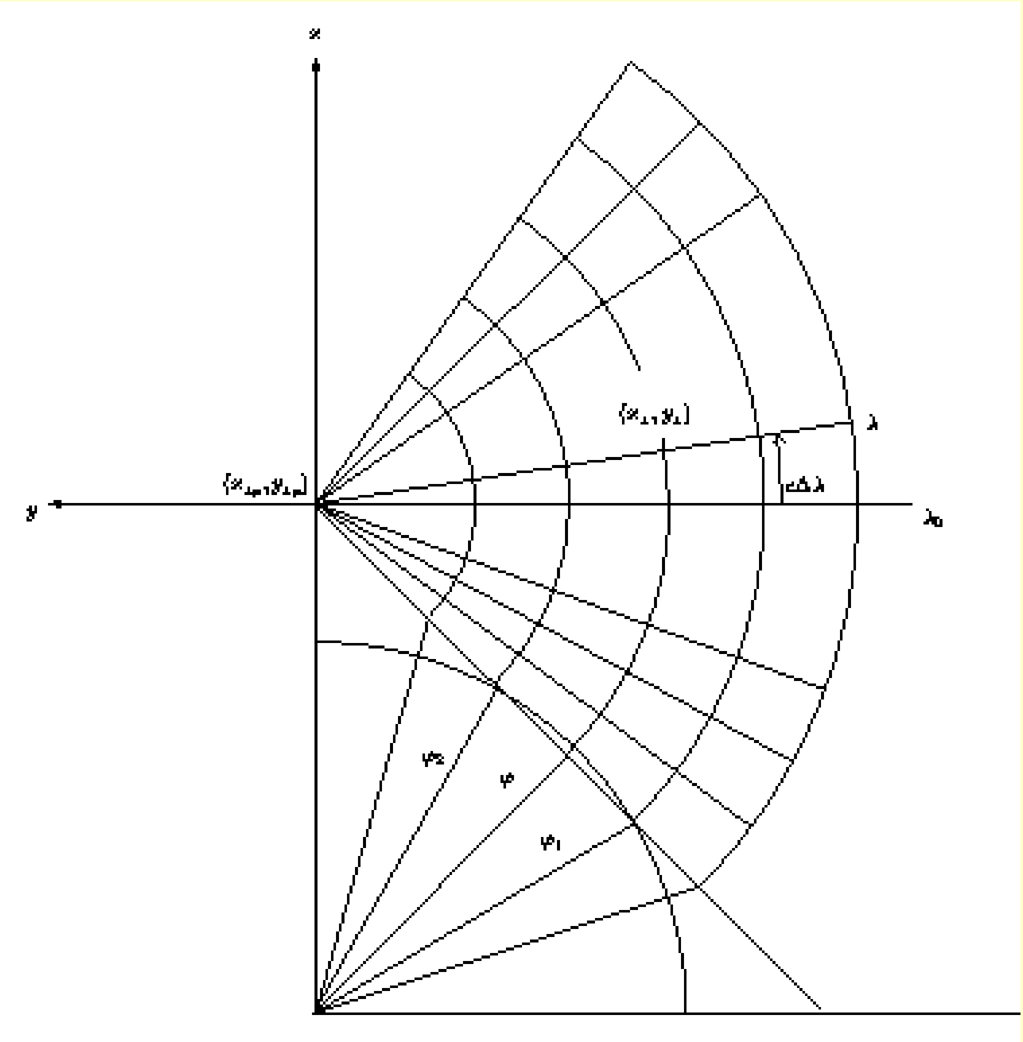
2) Lambert-conformal projection

$$c = \ln\left(\frac{\cos\varphi_1}{\cos\varphi_2}\right) / \ln\left\{\frac{\tan\left(\frac{\pi}{4}-\frac{\varphi_1}{2}\right)}{\tan\left(\frac{\pi}{4}-\frac{\varphi_2}{2}\right)}\right..$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_p + \dfrac{m}{c} a\cos\varphi\sin c\Delta\lambda \\ y_p - \dfrac{m}{c} a\cos\varphi\cos c\Delta\lambda \end{pmatrix}$$

$$m = \left(\frac{\cos\varphi}{\cos\varphi_1}\right)^{c-1}\left(\frac{1+\sin\varphi_1}{1+\sin\varphi}\right)^{c},$$

Normally, $\varphi_1 = \pi/6$, $\varphi_2 = \pi/3$

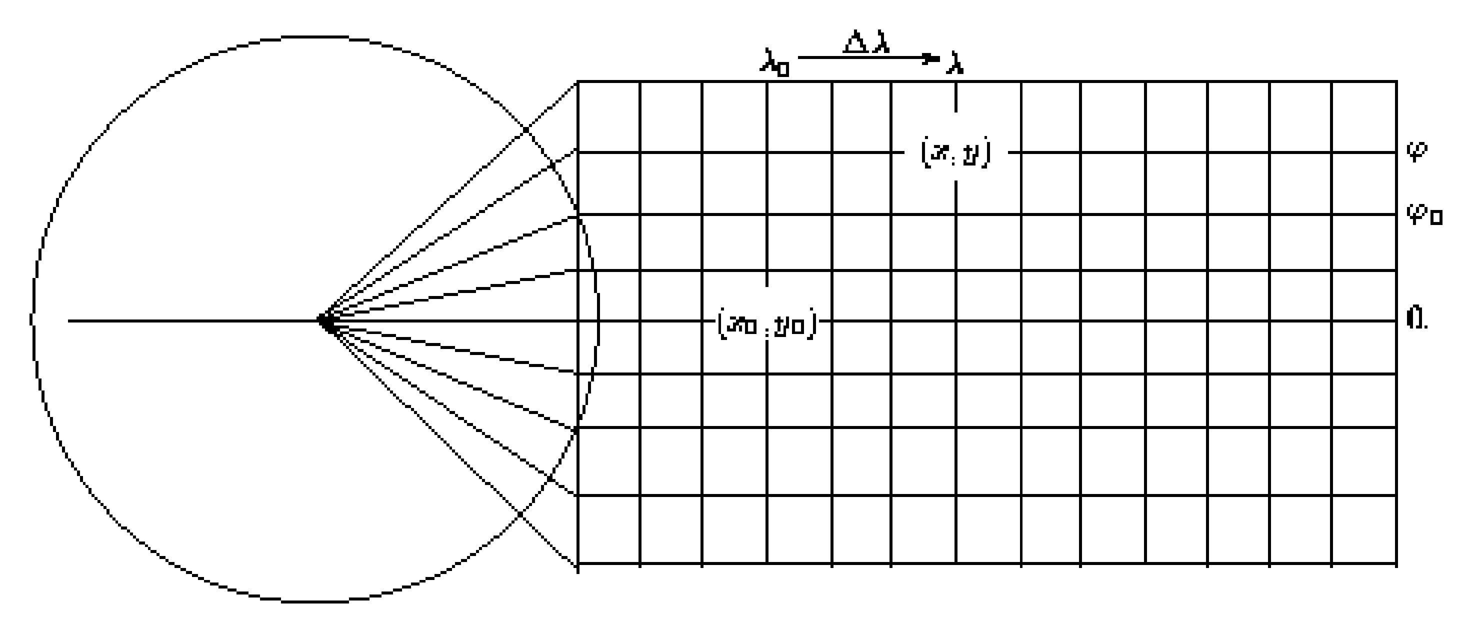


3) Mercator projection

$$c = 0,$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_o + a\cos\varphi_0 \Delta\lambda \\ y_o + a\cos\varphi_0 \ln\left(\dfrac{1+\sin\varphi}{\cos\varphi}\right) \end{pmatrix},$$

$$m = \frac{\cos\varphi_0}{\cos\varphi}.$$



Flux form equation

Continuity equation was

$$\frac{d\rho}{dt} + \rho m^2 \{ \frac{\partial}{\partial x}(\frac{u}{m}) + \frac{\partial}{\partial y}(\frac{v}{m}) \} + \rho \frac{\partial w}{\partial z} + \frac{2\rho w}{a} = prc \qquad (2.40)$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + m(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}) + w\frac{\partial}{\partial z} \qquad (2.41)$$

If we neglect the last term of *l.h.s.* of (2.40)by shallow assumption, we obtain

$$\frac{\partial\rho}{\partial t} + m(u\frac{\partial\rho}{\partial x} + v\frac{\partial\rho}{\partial y}) + w\frac{\partial\rho}{\partial z} + \rho m^2 \{ \frac{\partial}{\partial x}(\frac{u}{m}) + \frac{\partial}{\partial y}(\frac{v}{m}) \} + \rho \frac{\partial w}{\partial z}$$

$$= \frac{\partial\rho}{\partial t} + m^2 \{ \frac{\partial}{\partial x}(\frac{\rho u}{m}) + \frac{\partial}{\partial y}(\frac{\rho v}{m}) \} + \frac{\partial}{\partial z}(\rho w) = prc \qquad (2.42)$$

For arbitrary variable ϕ, from (2.41)

$$\frac{\rho}{m^2}\frac{d\phi}{dt}=\frac{\rho}{m^2}\frac{\partial\phi}{\partial t}+\frac{\rho u}{m}\frac{\partial\phi}{\partial x}+\frac{\rho v}{m}\frac{\partial\phi}{\partial y}+\frac{\rho w}{m^2}\frac{\partial\phi}{\partial z} \qquad (2.44)$$

From (2.42)

$$\frac{\phi}{m^2}\frac{\partial\rho}{\partial t}+\phi\{\frac{\partial}{\partial x}(\frac{\rho u}{m})+\frac{\partial}{\partial y}(\frac{\rho v}{m})\}+\frac{\phi}{m}\frac{\partial}{\partial z}(\frac{\rho w}{m})-\frac{\phi}{m^2}Prc=0 \qquad (2.45)$$

thus

$$\frac{\rho}{m^2}\frac{d\phi}{dt}=\frac{1}{m^2}\frac{\partial}{\partial t}(\rho\phi)+\frac{\partial}{\partial x}(\frac{\rho u\phi}{m})+\frac{\partial}{\partial y}(\frac{\rho v\phi}{m})+\frac{1}{m^2}\frac{\partial}{\partial z}(\rho w\phi)-\frac{\phi}{m^2}Prc \qquad (2.46)$$

or

$$\frac{\rho}{m}\frac{d\phi}{dt}=\frac{\partial}{\partial t}(\frac{\rho\phi}{m})+m\{\frac{\partial}{\partial x}(\frac{\rho u\phi}{m})+\frac{\partial}{\partial y}(\frac{\rho v\phi}{m})\}+\frac{\partial}{\partial z}(\frac{\rho w\phi}{m})-\frac{\phi}{m}Prc \qquad (2.47)$$

(2.31)～(2.33) become

$$\frac{\partial}{\partial t}(\frac{\rho u}{m}) + Adv.U + \frac{\partial p}{\partial x} = Crv.U + Cor.U + Dif.U \qquad (2.48)$$

$$\frac{\partial}{\partial t}(\frac{\rho v}{m}) + Adv.V + \frac{\partial p}{\partial y} = Crv.V + Cor.V + Dif.V \qquad (2.49)$$

$$\frac{\partial}{\partial t}(\frac{\rho w}{m}) + Adv.W + \frac{1}{m}(\frac{\partial p}{\partial z} + \rho g) = Crv.W + Cor.W + Dif.W \qquad (2.50)$$

Terms $Adv.U, Adv.V, Adv.W$: replace ϕ by u, v, w in
Terms $Crv.U$, $Crv.V$, $Crv.W$, $Cor.U$, $Cor.V$, $Cor.W$: multiply ρ/m to the terms of the equations in conformal projection

Prognostic equation for potential temperature is

$$\frac{d\theta}{dt} = \frac{\partial\theta}{\partial t} + ADV.\theta = \frac{Q}{C_p \pi} + Dif.\theta \qquad (2.51)$$

where

$$ADV.\theta = [m\{\frac{\partial}{\partial x}(\frac{\rho u\theta}{m}) + \frac{\partial}{\partial y}(\frac{\rho v\theta}{m})\} + \frac{\partial}{\partial z}(\frac{\rho w\theta}{m}) - \frac{\theta}{m}(\underline{Prc - \frac{\partial\rho}{\partial t}})]\frac{m}{\rho} \qquad (2.52)$$

underlined term is divergence.

Pressure equation

From state equation

$$\rho = \frac{p_0}{R\theta_m}\left(\frac{p}{p_0}\right)^{C_v/C_p}$$

$$\frac{\partial \rho}{\partial t} = -\frac{\rho}{\theta_m}\frac{\partial \theta_m}{\partial t} + \frac{1}{C_m{}^2}\frac{\partial p}{\partial t}$$

Here、C_m is the sound wave speed in case no liquid water

Combining continuity equation

$$\frac{\partial \rho}{\partial t} + m^2\{\frac{\partial}{\partial x}(\frac{\rho u}{m}) + \frac{\partial}{\partial y}(\frac{\rho v}{m})\} + \frac{\partial}{\partial z}(\rho w) = prc$$

Pressure equation is

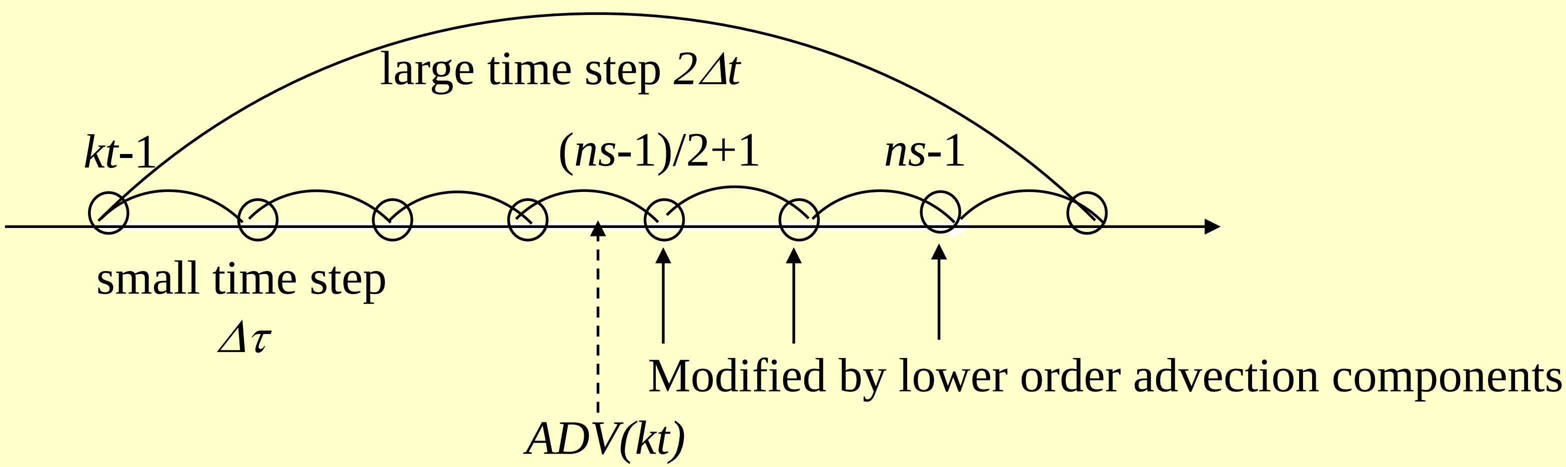
$$\frac{\partial p}{\partial t} = -C_m{}^2[m^2\{\frac{\partial}{\partial x}(\frac{\rho u}{m}) + \frac{\partial}{\partial y}(\frac{\rho v}{m})\} + \frac{\partial}{\partial z}(\rho w) - prc + \frac{\rho}{\theta_m}\frac{\partial \theta_m}{\partial t}]$$

c) Time-splitting of advection term

In order to enhance the computational robustness, advection terms of momentum and potential temperature are split at small time step

At the center of the Leapfrog time step, high-order advection terms are fully evaluated with the flux correction, and then second-order components are adjusted at each short time steps in the later half of the Leapfrog time integration

$$ADV = ADV(kt) - ADVL(kt) + ADVL^{\tau}$$



Advection terms are fully evaluated by higher order difference with flux correction

Δt = 40sec for 10km model; corresponds to Δt = 80 sec in RK2

Time splitting of gravity waves

$$\frac{\theta^{\tau+\Delta\tau} - \theta^{\tau}}{\Delta\tau} = -(ADV\theta - ADVL\theta + ADVL\theta^{\tau}) + \frac{Q}{c_p \pi} + dif.\theta$$

$$= ADVL\theta - ADVL\theta^{\tau} + \left[\frac{\partial\theta}{\partial t}\right], \qquad (19)$$

$$\frac{W^{\tau+\Delta\tau} - W^{\tau}}{\Delta\tau} + \frac{1}{mG^{\frac{1}{2}}}\frac{\partial P^{\beta}}{\partial z^{*}} + \frac{g}{mC_m{}^{2}}P^{\beta} = \frac{1}{m}BUOY^{\tau+\Delta\tau}$$

$$-(ADVW - ADVLW + ADVLW^{\tau} - RW)$$

$$+(1-\sigma)\frac{g}{mC_m{}^{2}}P^{\tau}. \qquad (20)$$

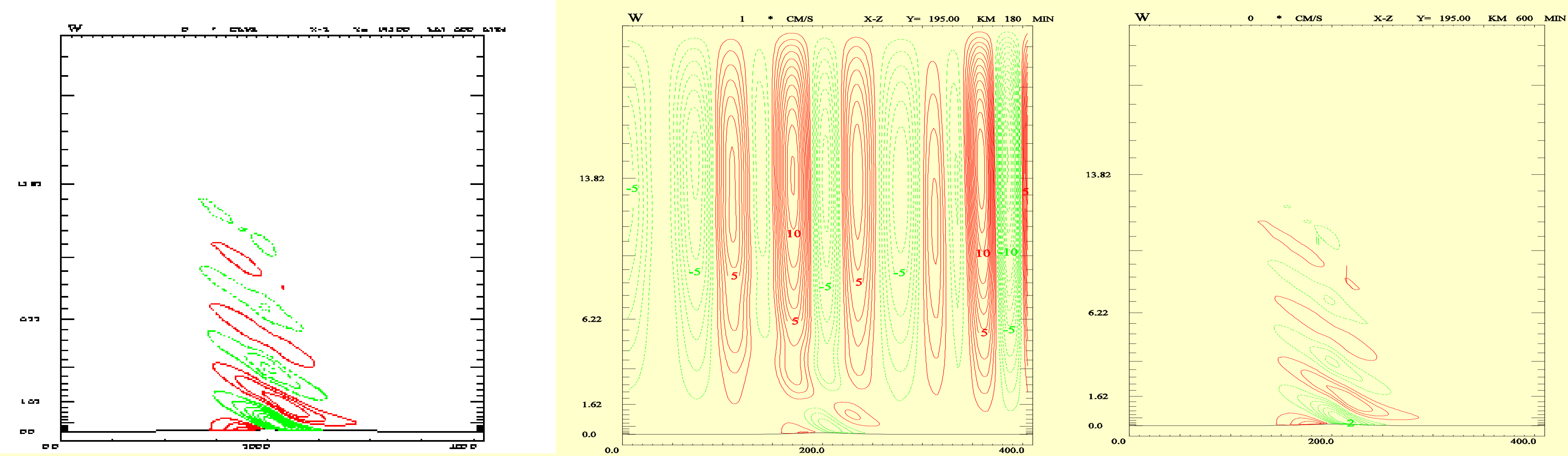
The tendency term in (19) is given by a tentative time integration in the cloud microphysical process.

Test of 3-D Linear Mountain waves

Uniform atmosphere with U=10m/s, N＝0.02/s

Bell-shaped mountain h=100m、a=30km with horizontal resolution 10km

$$Z_s(x,y) = \frac{h_m}{\{1+(\frac{x}{a})^2+(\frac{y}{a})^2\}^{\frac{3}{2}}},$$



Cross-section of vertical velocity without time-splitting of advection.

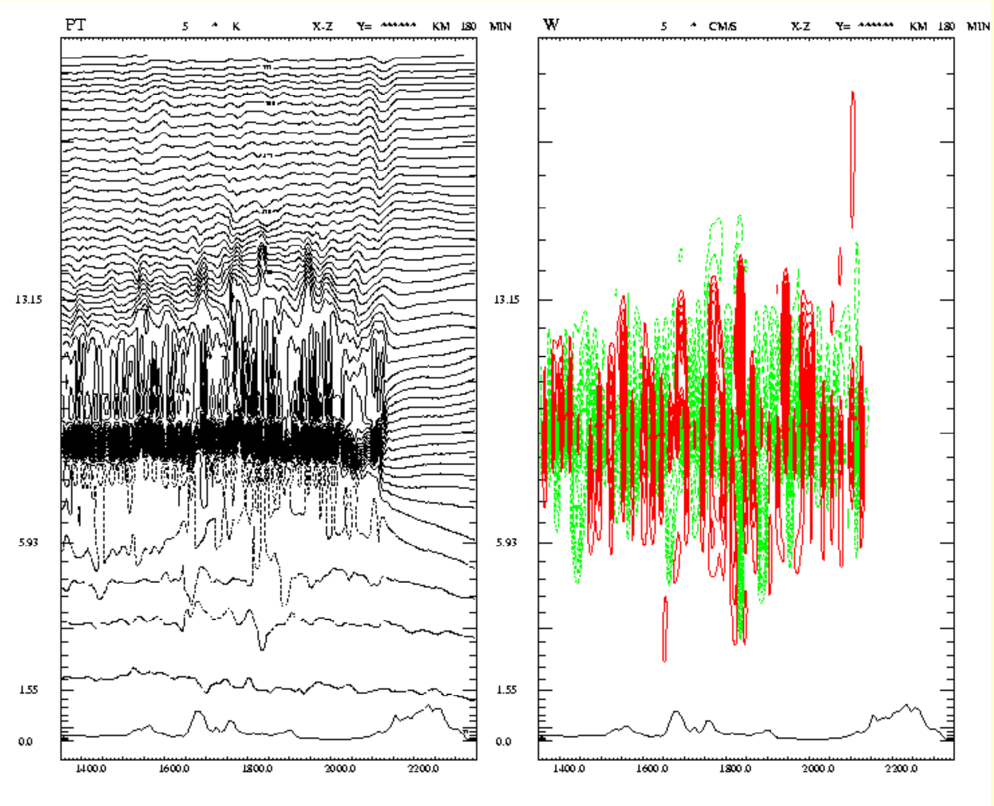
Left）Δt=30sec, t=10 hrs. Contour interval is 0.5 cm/s.

Center） Δt=40sec、t=2 hrs. Contour interval is1cm/s

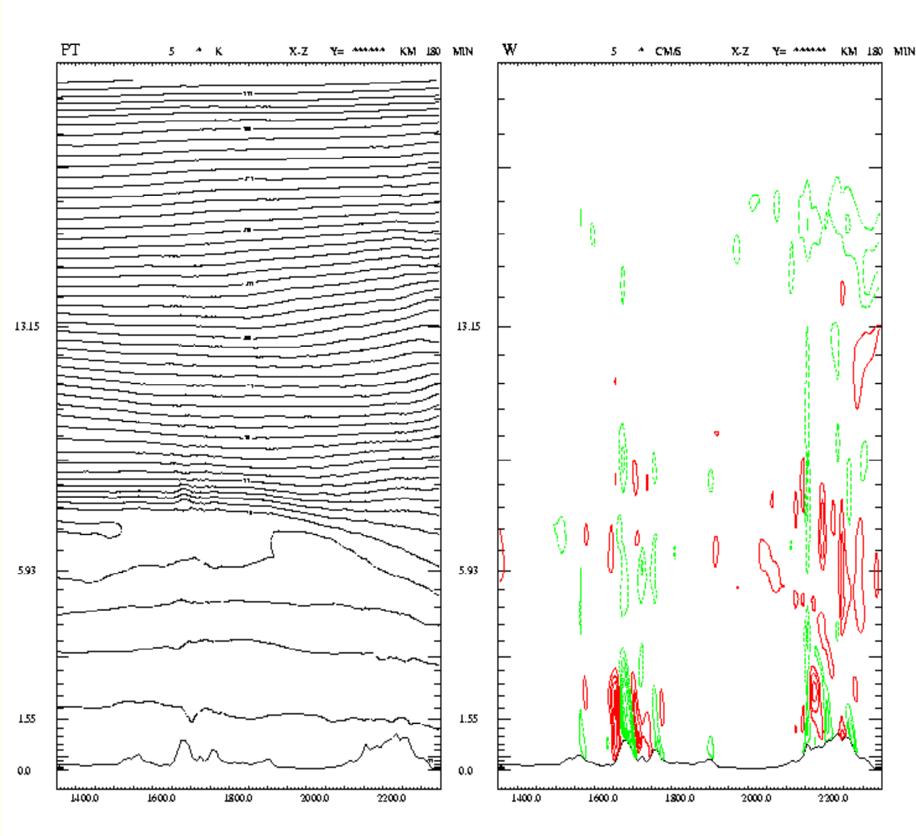
Right）Δt=50sec, t=10 hrs. Contour interval is 0.5 cm/s. With time-splitting of advection.

Case of 9 April 2001, 06UTC initial

Gravity waves sometimes become unstable when an inversion layer exists in strong wind environment.
Vertical cross-section of 3 hrs forecast of NHM with Δx=10km.
9 April 2001, 06UTC initial , Δt=40 sec.
Potential temperature and vertical wind.



Without time-splitting of advection



With time-splitting of advection

d) Divergence damping

Based on the idea of Wicker and Skamarock (1992), while acts on the flux form total divergence

$$\frac{U^{\tau+\Delta\tau}-U^{\tau}}{\Delta\tau}+\frac{\partial P^{\tau}}{\partial x}+\frac{\partial G^{\frac{1}{2}}G^{13}P^{\tau}}{G^{\frac{1}{2}}\partial z^{*}}=-(ADVU+RU)+\underline{\frac{\partial}{\partial x}\alpha_{H}DIVT},$$

$$\frac{V^{\tau+\Delta\tau}-V^{\tau}}{\Delta\tau}+\frac{\partial P^{\tau}}{\partial y}+\frac{\partial G^{\frac{1}{2}}G^{23}P^{\tau}}{G^{\frac{1}{2}}\partial z^{*}}=-(ADVV+RV)+\underline{\frac{\partial}{\partial y}\alpha_{H}DIVT},$$

$$\frac{W^{\tau+\Delta\tau}-W^{\tau}}{\Delta\tau}+\frac{1}{mG^{\frac{1}{2}}}\frac{\partial P^{\beta}}{\partial z^{*}}+\frac{g}{mC_{m}^{2}}P^{\beta}=\frac{1}{m}BUOY-(ADVW-RW)+\underline{\frac{\partial}{\partial z}\alpha_{Z}DIVT}$$

where

$\alpha_H = 0.06 * (\Delta x)^2/\Delta\tau$

$\alpha_{Z(}(kz) = 0.05 * (\Delta z(kz))^2/\Delta\tau$

e) Direct evaluation of buoyancy

The buoyancy term *BUOY* is defined by

$$BUOY \equiv \sigma \frac{\rho G^{\frac{1}{2}} \theta_m'}{\theta_m} g + (1-\sigma)(\overline{\rho} - \rho) g G^{\frac{1}{2}} \quad (24)$$

σ=0: density perturbation, σ=1: potential temperature perturbation

vertical momentum equation

$$\frac{\partial W}{\partial t} + \frac{1}{mG^{\frac{1}{2}}} \frac{\partial P}{\partial z^*} + \sigma \frac{P}{mC_m{}^2} = \frac{1}{m} BUOY - ADVW + RW.$$

In case of $\sigma = 0$, BUOY includes pressure perturbation, which has to be treated implicitly.

Implicit treatment

Independent of σ, pressure perturbation term is treated implicitly as

$$\delta_\tau W + \frac{1}{mG^{\frac{1}{2}}}\frac{\partial P^\beta}{\partial z^*} + \frac{g}{mC_m{}^2}P^\beta = \frac{1}{m}BUOY - (ADVW - RW) + (1-\sigma)\frac{g}{mC_m{}^2}P. \qquad (2.2.29)$$

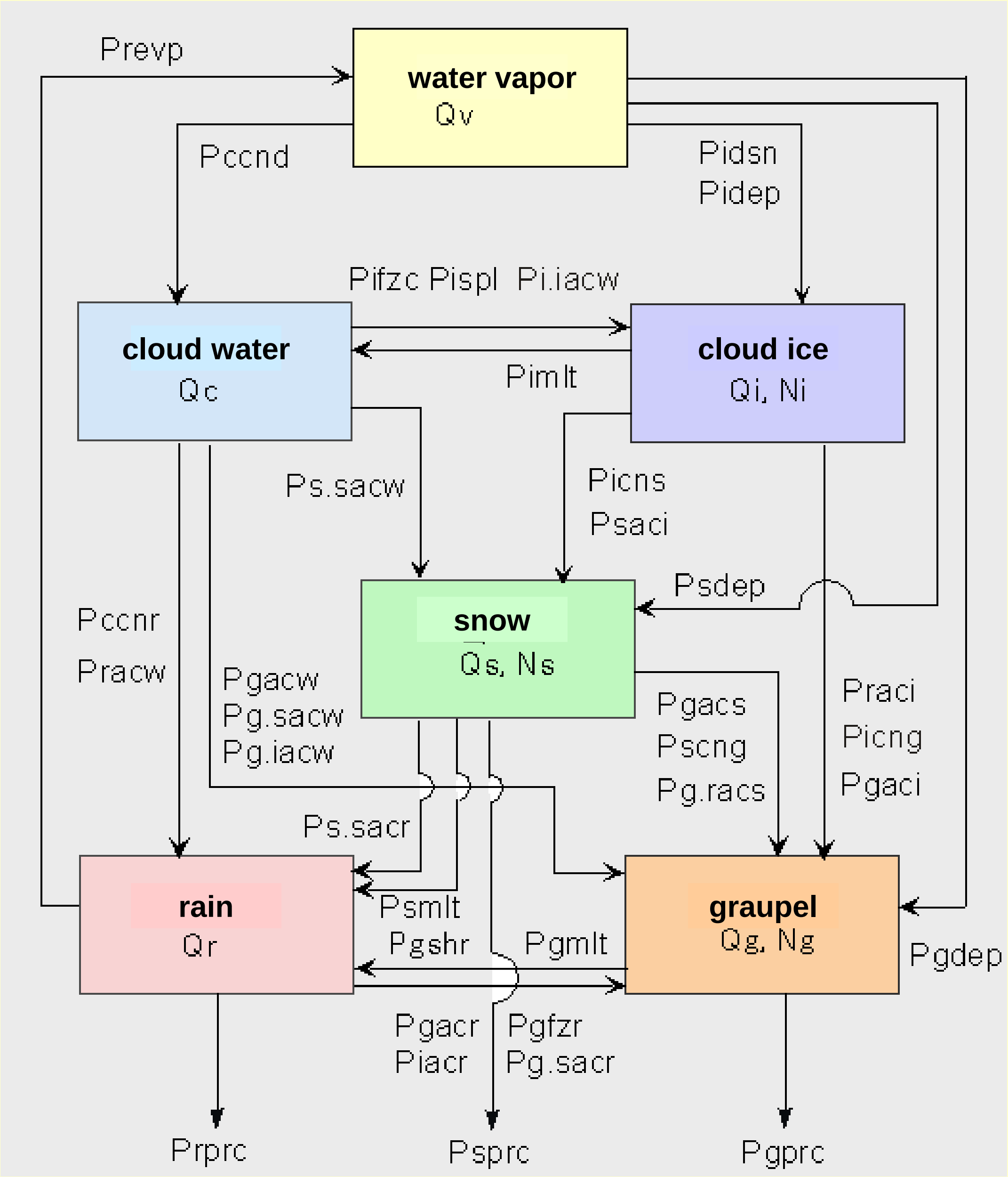
The last term in *r.h.s.* is evaluated in short time step.
Upper and lower boundary condition of pressure equation is

$$(\frac{1}{mG^{\frac{1}{2}}}\frac{\partial}{\partial z^*} + \frac{g}{mC_m{}^2})P^\beta = -(ADVW - RW) + \frac{1}{m}\{BUOY + (1-\sigma)\frac{g}{C_m{}^2}P\}. \qquad (2.2.39)$$

To split gravity waves, diagnosis of density is required in each short time step.

Cloud Microphysics

- Typical prognostic variables
  - water vapor, cloud water, rain, cloud ice, and graupel (Qv, Qc, Qr, Qi, Qs, Qg)
  - number density of cloud ice, snow and graupel (Ni, Ns, Ng)



Tetens' formula for
saturated vapor pressure [hPa]

$$e_s = 6.11 \times 10^{\frac{7.5 \times t}{237.7 + t}}$$

Bulk cloud microphysics

In bulk method, the size distribution function of water substance is expressed by the inverse exponential function of the particle diameter *D*.

$$N(D) = N_0 e^{-\lambda D}$$

Fall-out terminal velocity of particle is given as a power function of *D* by the Stokes' law in the form of

$$V(D) = aD^b$$

Variable Qx(kg/kg) Nx(m^{-3})	Size distribution $Nx(D)$ (m^{-4})	Fall velocity Udx(m/s)	Density ρ_x(kg/m^3)
Qr	$Nr(D) = Nr_0 \exp(-\lambda D)$ $Nr_0 = 8 \times 10^6$	$a_r Dr^{b_r} \left(\frac{\rho_0}{\rho}\right)^{1/2}$ $a_r = 842$ $b_r = 0.8$	$\rho_w = 1 \times 10^3$
Qs Ns	$Ns(D) = Ns_0 \exp(-\lambda D)$ $(Ns_0 = 1.8 \times 10^6)$	$a_s Ds^{b_s} \left(\frac{\rho_0}{\rho}\right)^{1/2}$ $a_s = 17$ $b_s = 0.5$	$\rho_s = 8.4 \times 10$ $r_{s0} = r_0 = 75\,\mu$m $m_{s0} = (4\pi/3)\rho_s r_{0s}^3$
Qg Ng	$Ng(D) = Ng_0 \exp(-\lambda D)$ $(Ng_0 = 1.1 \times 10^6)$	$a_g Dg^{bg} \left(\frac{\rho_0}{\rho}\right)^{1/2}$ $a_g = 124$ $b_g = 0.64$	$\rho_g = 3 \times 10^2$ $r_{g0} = r_0 = 75\,\mu$m $m_{g0} = (4\pi/3)\rho_g r_{0g}^3$
Qc	mono $Di = \left(\frac{6\rho Qc}{\pi \rho_w Nc}\right)^{1/3}$ $Nc = 1 \times 10^8$m^{-3}	$a_c Dc^{bc}$ $a_c = 3 \times 10^7$ $b_c = 2.0$	$\rho_c = 1.0 \times 10^3$
Qi Ni	mono $Di = \left(\frac{6\rho Qi}{\pi \rho_i Ni}\right)^{1/3}$	$a_i Di^{bi} \left(\frac{\rho_0}{\rho}\right)^{0.35}$ $a_i = 7 \times 10^2$ $b_i = 1.0$	$\rho_i = 1.5 \times 10^2$ $m_{i0} = 1 \times 10^{-12}$kg

Bulk cloud microphysics

The mass-weighted mean velocity is obtained by

$$\overline{V} = \frac{\int \frac{\pi}{6}\rho_w D^3 V(D) N(D) dD}{\int \frac{\pi}{6}\rho_w D^3 N(D) dD} = \frac{\int D^3 a D^b e^{-\lambda D} dD}{\int D^3 e^{-\lambda D} dD} = \frac{a\Gamma(4+b)}{6\lambda^b}$$

Here Γ(z) is the Gamma function

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$$

By Euler's partial integration, the Gamma function has the following characteristic

$$\Gamma(z) = \int_0^\infty t^{z-1}(-e^{-t})' dt = \left[-t^{z-1}e^{-t}\right]_0^\infty + (z-1)\int_0^\infty t^{z-2} e^{-t} dt$$

$$= (z-1)\Gamma(z-1)$$